Approximating Invariants through Polynomial Functors

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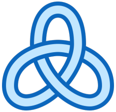
Math 518 Final Presentation

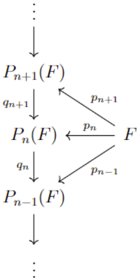
Roadmap



Geometric Motivation/History - 1 Slides

Classifying spaces by invariants

Chain complexes and algebraic topology



Polynomial Functors - 7 Slides

What is a functor?

Polynomial approximation: the goal

Polynomial approximation: the construction

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Classifying Spaces up to Continuous Deformations

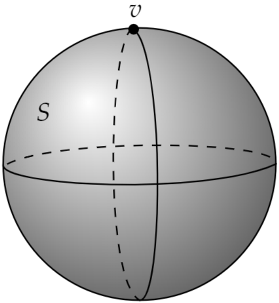

Figure: A sphere with a specified vertex.

Δ-chain for Sphere: [3]

$$
\begin{array}{ccc}
\vdots & & \\
\downarrow & & \\
0 & & \text{dim} = 3 \\
\downarrow & & \\
\mathbb{Z}_S & & \text{dim} = 2 \\
\downarrow & & \\
0 & & \text{dim} = 1 \\
\downarrow & & \\
\mathbb{Z}_v & & \text{dim} = 0
\end{array}
$$

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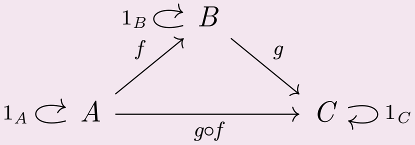
Categories

Def^{n} [6, Defn 1.1.1]: Categories

A category, $\mathcal{C}$, consists of a collection of objects and maps between objects which can be composed.



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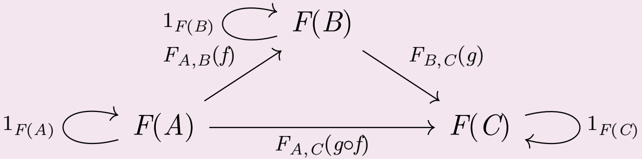
Functors

Def^{n} [6, Defn 1.3.1]: Functors

A functor $F : \mathcal{A} \to \mathcal{B}$ between categories is a function F on objects and functions $F_{A,B}$ on maps such that



Polynomial Approximations

Goal:

How can we simplify and study functors of the form $F : \mathcal{B} \to \mathsf{Ch}(\mathsf{Ab})$?

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Solution: Introduce a chain of simpler functors $P_n(F) : \mathcal{B} \to \mathsf{Ch}(\mathsf{Ab})$, $n \in \mathbb{N}$, which "approximate" F in the limit [4]:

$$\begin{array}{c} F \xrightarrow{p_{n+1}} P_{n+1}(F), \quad F \xrightarrow{p_n} P_n(F), \quad F \xrightarrow{p_{n-1}} P_{n-1}(F), \quad F \xrightarrow{p_0} P_0(F) \\ \cdots \longrightarrow P_{n+1}(F) \xrightarrow{q_{n+1}} P_n(F) \xrightarrow{q_n} P_{n-1}(F) \longrightarrow \cdots \xrightarrow{q_1} P_0(F) \end{array}$$

Construction: Cross-Effects

Question:

What does it mean for a functor $F : \mathcal{B} \to \mathsf{Ch}(\mathsf{Ab})$ to be "simple"?

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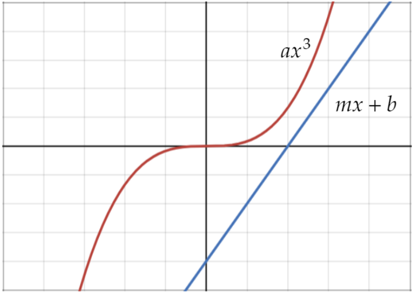

Figure: Cubic and shifted linear plots.

Construction: Cross-Effects

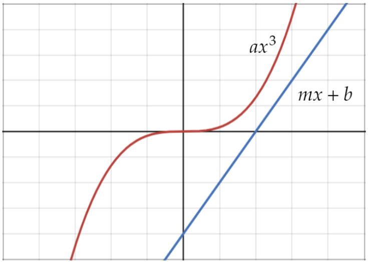

Figure: Cubic and shifted linear plots.

Measuring Defect

For $f\colon \mathbb{R} \to \mathbb{R}$, the defect to being polynomial can be measured by using cross-effects, such as

$$\mathrm{cr}_1(f)(x) = f(x) - f(0)$$

and

$$\mathrm{cr}_2(f)(x, y) = \mathrm{cr}_1(f)(x + y) - \mathrm{cr}_1(f)(x) - \mathrm{cr}_1(f)(y)$$

Construction: Cross-Effects

Remark: We can generalize the definition to functors in a natural way using the implicit definition [2]

$$\mathsf{cr}_1(F)(A) \oplus F(0) \cong F(A)$$

and

$$\mathsf{cr}_2(F)(A, B) \oplus \mathsf{cr}_1(F)(A) \oplus \mathsf{cr}_1(F)(B) \cong \mathsf{cr}_1(F)(A \oplus B)$$

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Example: "f(x)=x+a"

Let $A \in \mathsf{Ab}$ and let $T_A : \mathsf{Ab} \to \mathsf{Ch}(\mathsf{Ab})$ be given by $T_A(B) = \cdots \to 0 \to 0 \to A \oplus B$. Then

$$\mathsf{cr}_1(T_A)(B) \cong \cdots \to 0 \to 0 \to B$$

and

$$\mathsf{cr}_2(T_A)(B,C) \cong \cdots \to 0 \to 0 \to 0$$

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Construction: Polynomial Functors

Defn: Zeroth Polynomial Approximation [4]

$P_0(F)$ is given by resolving F with respect to cross-effects, forming the inclusion

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathsf{cr}_1^3(F) & \longrightarrow & \mathsf{cr}_1^2(F) & \longrightarrow & \mathsf{cr}_1(F) \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & F \end{array}$$

into F isolated in degree 0, and then "totalizing".

If $F = \cdots \to 0 \to 0 \to F_0$, then this becomes the augmented complex:

$$\cdots \longrightarrow \mathsf{cr}_1^3(F_0) \longrightarrow \mathsf{cr}_1^2(F_0) \longrightarrow \mathsf{cr}_1(F_0) \longrightarrow F_0$$

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Example: For $T_A : \mathrm{Ab} \to \mathrm{Ch}(\mathrm{Ab})$,

$$P_0(T_A)(B) = \cdots \to B \xrightarrow{1_B} B \xrightarrow{0} B \xrightarrow{i} A \oplus B$$

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After contracting:

$$P_0(T_A)(B) \simeq \cdots 0 \to 0 \to 0 \to A$$

Key Takeaways:

- ☐ Algebraic invariants help classify spaces
- ☐ Algebraic invariants are rich in properties
- ☐ Invariants can be approximated in terms of Taylor series-like methods
- ☐ These approximations can be constructed concretely for chain complexes

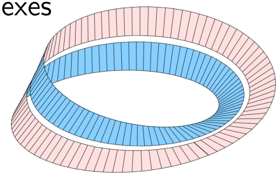
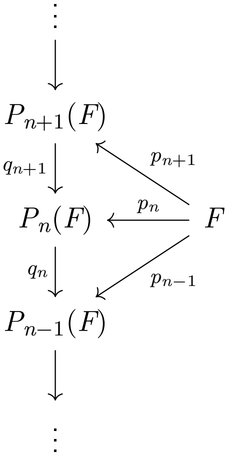


Figure: Mobius strip diagram [1].



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